Solitaire Clobber

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Abstract

Solitaire Clobber is the one-player variant of Clobber, where the goal is to remove as many stones as possible from the board by alternating white and black moves.

In this paper, we see some values and conjectures about the reducibilty of some Solitaire Clobber configuration, played on a board or in some classes of graph. We also propose a short introduction to the 2-players Clobber game strategie, even if there is not much known about it yet. We finally expose some opened problems concerning Clobber in all of its variants.

1 Introduction

Clobber is a two-player combinatorial board game introduced by Albert et al. at the 2002 Dagstuhl seminar on algorithmic and Combinatorial Game Theory. It is played with black and white stones occuping some subset of the squares of an $n \times m$ cherckerboard (i.e. grid graph) but the rules easily generalize to play on an arbitrary graph. Each player move alternately by picking up one of their own stones and *clobbering* an adjacent opponent's stone. The clobbered stone is removed from the board, and replaced by the stone that was moved. The first player who is unable to move on his turn loses.

In [4], Demaine et al. introduce a one-player variant, Solitaire Clobber. The goal is to remove as many stones as possible from the board by alternating white and black moves. We say a configuration is k-reductible if we can end it up with k irremovable stones.

Starting from their works, more investigations have been performed. While Dorbec et al. play Clobber on graphs [5], Albert et al. try to study Clobber strategies and position estimations [1, 2, 3]. There have also been some attempts to organize computer Clobber tournaments [10], where methods such as Monte Carlo ones have been tested.

This paper is organized as follow: Section 2, we try to make a round of the knowledge showed in [4], while we see reducibility of some configurations played on graphs in Section 3. In Section 4, we finally analyse some conjectures on 2-player clobber strategies. We conclude with some open problems in section 5.

2 Solitaire Clobber

In their paper [4], Demaine et al. study the reducibility of some clobber board configurations.

We say a stone is *matching* if it has the same colour as the square it occupies on the underlying checkerboard; otherwise we say it is *clashing*. In a *checkerboard configuration*, all stones are matching, i.e., the white stones occupy white squares and the black stones occupy black squares. And in a *rectangular* configuration, the stones occupy exactly the squares of some rectangular region on the board. Usually, Clobber starts from a rectangular checkerboard configuration, and White moves first.

Obviously, 1-reducibility can only be possible if half of the stones are white, and half of the stones are black. But even then it might not be possible.

In one-dimensional Solitaire Clobber (i.e. the board consists of a single row of alternating white and black stones) we can usually not expect 1-reducibility. We see that the checkerboard configuration can be reduced to $\lceil n/4 \rceil + \{1 \text{ if } n \equiv 3 \pmod 4\}$ stones, no matter who move first, by slipping the initial configuration A_n into $\lceil n/4 \rceil$ substings, n being the number of stones.

Theorem 2.1 *For $n \geqslant 1$, the configuration A_n can be reduced to $\lceil n/4 \rceil + \{1 if n \equiv 3 (\mathrm{mod}\, 4)\}$ stones by an alternating sequence of moves, no matter who is to move first.*

Grossman independantly showed that this bound is best possible even if we do not have to alternate between white and black moves (see [6]).

Theorem 2.2 *Let $n \geqslant 1$. Even if we are not restricted to alternate white and black moves, the configuration A_n can not be reduced to fewer than $\lceil n/4 \rceil + \{1 if\, n \equiv 3 (\mathrm{mod}\, 4)\}$ stones.*

Demaine et al. prove in [4] that a necessary condition for a Clobber position to be 1-reducible is that the number of stones plus the number of clashing stones cannot be a multiple of three. We denote by $\delta(C)$ this number for a configuration C.

Theorem 2.3 *For a configuration C, $\delta(C) (\mathrm{mod}\, 3)$ does not change after an arbitrary move sequence.*

Corollary 2.4 *A configuration C with $\delta(C) \equiv 0 (\mathrm{mod}\, 3)$ is not 1-reducible.*

This condition is also sufficient for truly two-dimentional rectangular $n \times m$ chekerboards (i.e. with $n, m \geqslant 2$). If this condition is not true, then the board is 2-reducible. To prove that, we have to distinguish the four followingseparate cases based on the paritiess of n and m. For each case, they follow the same strategy, eliminating the stones on the board from top to bottom, two rows at a time.

Theorem 2.5 *For $n, m \geqslant 2$, a rectangular checkerboard configuration with n rows and m columns is 2-reducible if $nm \equiv 0 (\mathrm{mod}\, 3)$, and is 1-reducible otherwise.*

We can also see that in general it is NP-complete to decide whether an arbitraty configuration is 1-reducible. The proof is by reduction from the Hamiltonian circuit problem in grid graphs. Itai and al. proved that GRID-HAMILTONICITY (i.e. decide whether a given grid graph has a Hamiltonian circuit) is NP-complete (see [8]).

3 Playing Clobber on graphs

Duchêne et al. study the reducibility of a variant of Solitaire Clobber played on graph in [5]. In this new variant named *SC2*, it is not necessary to altenate white and black moves. In this way, we see an initial configuration does not need to have the same number of white stones and black stones to be 1-reducible. The authors give algorithms for different classes of graph, such as paths, cycles, trees, cliques, clique products and hamming graphs.

The theorems on reducibility of one-dimensional and arbitraty grid-graph configuration exposed by Demaine et al. in [4] stay true in SC2.

We fisrt see reducibility for string and path, first studying 1-reducible ones.

Given a graph $G = (V, E)$, we say a vertice $v \in V$ is *white* (resp. *Black*) if it is occupied by a white stone (resp. a black stone). We say a vertice is *empty* if no stone is placed on it.

We consider paths and cycles as strings on the set $\{X, O\}$ where X (resp. O) represent a black stone (resp. a white stone). For example, ●○●○○● is the string $XOXOOX$. Duchêne et al. then show that 1-reducible strings are one of the following: strings of length 1 (i.e. X or O) and strings like $\{XO^{\star}X^{\star}O\}$, or $\{OX^{\star}O^{\star}X\}$ by symmetry. For a graph configuration G (path or cycle), we search every 1-reducible sub-strings of G. We build the intersection graph $\Gamma = (V, E)$ where evey vertice is one of these sub-strings. Two vertex of Γ are connected if their corresponding sub-strings have a common vertice in G. The reducibility of G is resolved by searching the dominating stable set in Γ. This algorithm can be resolved in polynomial time (see [9]).

Because the number of 1-reducible sub-trees contained in an initial tree is not necessary polynomial, this kind of algorithm can not be applied to trees. Duchêne et al. propose a different algorithm.

Theorem 3.1 *Given an arbitraty configuration of SC2 on a tree, we can determine its best possible reducibility in a quadratic time.*

Proposition 3.2 *Given a configuration on a clique K_n with at least one stone of each colour, there is a move sequence that allow ending up with one stone.*

Theorem 3.3 *If there is at least one stone of each colour, the clique products $K_p \square K_q$ are 1-reducible if $p \geqslant 3$ or $q \geqslant 3$.*

Most of cliques and clique products configuration in SC2 are 1-reducible. Concerning Hamming graph, there exists a proof that they are 1-reducible if they contain at least one clique of size 3.

Theorem 3.4 *Given an initial configuration on a hypercude Q_n, Q_n is 2-reducible if there is at least one stone of each colour.*

Reader can refer to [5] for detailled proves of those theorems.

4 2-Players Clobber strategies

Since they developped Clobber, Albert et al. have been studying 2-players strategies [1].

The notation and game values studied here are developped, in depth, in [2] and [3].

Clobber is an *all-small* game [2, 3], as there is no position in which one player can move and the other cannot. The game values are all *infinitesimals.*

The $1 \times n$ and $2 \times n$ positions considered are primarily of theretical interest but do not generate interesting play between two players for the most part of them. By contrast, the $m \times n$ checkerborad, with white pieces starting on white squares and black pieces starting on black squares are, currently, mathematically intractable.

Here is some game values of simple positions.

$$\circ = \bullet = 0$$

$$\circ\bullet = *$$

$$\circ\bullet\bullet = \{0|\circ\bullet\} = \{0|*\} = \uparrow$$

And here are some conjectures about $1 \times n$ and $2 \times n$ positions.

Lemma 4.1 $\bullet^m\circ^n = 0$ *for* $m, n \geqslant 2$;
and $\bullet\circ^n = (n-1).\uparrow + n.*$.

Conjecture 4.2 $(\bullet\circ)^n$ *is a first player win for* $n \neq 3$.

Conjecture 4.3 $(\bullet\bullet\circ)^n = \lfloor (n+1)/2 \rfloor . \uparrow$.

Now we consider the $2 \times n$ position $(\genfrac{}{}{0pt}{}{\circ}{\bullet})^n = \overbrace{\genfrac{}{}{0pt}{}{\circ\circ\circ}{\bullet\bullet\bullet}\cdots\genfrac{}{}{0pt}{}{\circ}{\bullet}}$ with n columns

Lemma 4.4 *If n is even then* $(\genfrac{}{}{0pt}{}{\circ}{\bullet})^n = 0$

Conjecture 4.5 *For n odd* $(\genfrac{}{}{0pt}{}{\circ}{\bullet})^n$ *is a first player win.*

The game values for Clobber, in general, do not have nice canonical forms. *Atomic weights* are introduced to avoid complicated infinitesimals. The authors finally show that Clobber is NP-hard by determining the atomic weight for an arbitrary graph that has just one white token, and the rest are black. The proof is done by induction.

Despite all these investigations, NP-hardness of clobber played on a grid-graph is still an open problem.

5 Conclusion

Since [4], we see many investigations have been done on Clobber and some of its variants.

For the one-player variant, Solitaire Clobber, some structural parameters have been identified, which influence reducibility. But there is still more works to do, like characterizing the 1-reducible hypercubes (every hypercubes are not 1-reducible).

Still, there is very little known about Clobber strategies or position estimations, and the computation of CGT game values still in its preliminary stages.

Finally, some problems are still left opened, as the NP-hardness of clobber played on a grid-graph, or the conjectures on $1 \times n$ and $2 \times n$ configurations, as said in [7].

References

[1] M. H. Albert, J. P. Grossman, R. J. Nowakowski, and D. Wolfe. An introduction to clobber. *INTEGERS*, 5(2), 2004.

[2] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways*. A K Peters, Ltd., 2001.

[3] J. H. Conway. *On Numbers and Games*. A K Peters, Ltd., 2001.

[4] E. D. Demaine, M. L. Demaine, and R. Fleischer. Solitaire clobber. *Theoretical Computer Science*, 313(3):325–338, February 2004.

[5] P. Dorbec, E. Duchêne, L. Faria, and S. Gravier. Solitaire clobber played on graphs.

[6] J. P. Grossman. Private communication. March 2002.

[7] R. K. Guy and R. J. Nowakowski. Unsolved problems in combinatorial games. *MSRI Publications*, 42, 2002.

[8] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.

[9] W. F. Klostermeyer and E. M. Eschen. Perfect codes and independant dominating sets. *Congr. Numerantium*, 142:7–28, 2000.

[10] J. Willemson and M. Winands. Mila wins clobber tournament. 2006.